

THE APPROXIMATION BY PARTIAL SUMS OF FOURIER SERIES

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1. Let $f(x)$ be a continuous function having the period 2π . If this function satisfies a Lipschitz condition of order α ($0 < \alpha \leq 1$), that is to say if

$$|f(x+h) - f(x)| = O(|h|^\alpha)$$

uniformly in x as $h \rightarrow 0$, we shall say briefly that f belongs to Lip α . Following a notation already used (see Zygmund [4]⁽¹⁾) we shall say that f belongs to lip α , $0 < \alpha < 1$, if $|f(x+h) - f(x)| = o(|h|^\alpha)$ uniformly as $h \rightarrow 0$.

It is a classical result of Lebesgue (see Zygmund [5, p. 61]; hereafter this book will be denoted by T.S.) that if f belongs to Lip α and if s_n denotes the n th partial sum of the Fourier series of f , then, uniformly

$$(1) \quad |s_n(x) - f(x)| = O(n^{-\alpha} \log n)$$

and it is known that the logarithm in the second member cannot be omitted.

The question whether the last relation can be sharpened by the suppression of the logarithm if, in addition to the hypothesis that f belongs to Lip α , we suppose also that f is of bounded variation seems to have been overlooked and it is the main purpose of this paper to give the solution of this problem. It will be sufficient to consider the case $0 < \alpha < 1$ since if $\alpha = 1$ it is well known that the relation (1) can not be sharpened (see T.S. pp. 61–62, examples 6 and 8).

Let $a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be the Fourier series of a continuous function of bounded variation, $f(x)$ of period 2π . It is clear that such a function can not be monotonic in $(0, 2\pi)$ since $f(0) = f(2\pi)$; but we shall say that $f(x)$ belongs to the monotonic type if the Fourier-Stieltjes series

$$d(f + Cx) \sim C + \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx)$$

is, for some C , the Fourier-Stieltjes series of a monotonic function; in other words if there exists a constant C such that $f(x) + Cx$ is either never decreasing or never increasing in $(-\infty, +\infty)$.

Our results can now be stated in the following two theorems:

THEOREM I. *If the continuous function $f(x)$ of period 2π belongs to Lip α ($0 < \alpha < 1$) and is of monotonic type, then*

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(¹) Numbers in brackets refer to the literature cited at the end of the paper.

$$|s_n(x) - f(x)| = O(n^{-\alpha})$$

uniformly in x , where $s_n(x)$ is the n th partial sum of the Fourier series of f .

THEOREM II. *The preceding theorem is no longer true if instead of supposing that $f(x)$ is of monotonic type we suppose that $f(x)$ is of bounded variation. More precisely, for every positive α less than 1 there exists a continuous function $\phi(x)$ of bounded variation belonging to $\text{Lip } \alpha$ and such that for arbitrarily large values of n*

$$\max |s_n(x) - \phi(x)| > \frac{\log n}{\omega(n) \cdot n^\alpha},$$

$\omega(n)$ being a function increasing infinitely with n , as slowly as we please.

Proof of Theorem I. Suppose, to fix the ideas, that there is a constant C such that $F(x) = f(x) + Cx$ is increasing. Denoting as usual by $D_n(t)$ the Dirichlet's kernel $\sin((2n+1)t/2) (2 \sin t/2)^{-1}$ we have

$$\pi[s_n(x) - f(x)] = \int_0^\pi [f(x+t) + f(x-t) - 2f(x)] D_n(t) dt.$$

Since $f \in \text{Lip } \alpha$, it is obvious that

$$\int_0^\eta [f(x+t) + f(x-t) - 2f(x)] D_n(t) dt = O(n^{-\alpha})$$

uniformly in x , if $\eta = O(1/n)$. To study the integral from η to π , we observe that it is equal to

$$(2) \quad \int_\eta^\pi [F(x+t) + F(x-t) - 2F(x)] D_n(t) dt$$

and we split it into integrals of the type $\int_{\pi/2^k}^{\pi/2^{k-1}}$ where k runs from 1 to the integral part of $\log n / \log 2$. By the second mean value theorem, the integral in the interval $(\pi/2^k, \pi/2^{k-1})$ is equal to

$$\frac{1}{2 \sin(\pi/2^{k+1})} \int_{\pi/2^k}^\xi [F(x+t) + F(x-t) - 2F(x)] \sin\left(\frac{2n+1}{2}t\right) dt$$

$$\left(\frac{\pi}{2^k} < \xi < \frac{\pi}{2^{k-1}}\right)$$

and since $F(x+t) + F(x-t) - 2F(x)$ is an increasing function of t an application of the second mean value theorem shows that this expression is

$$O\left[\left(\frac{2^k}{\pi}\right)\left(\frac{\pi}{2^k}\right)^\alpha \frac{1}{n}\right] = O\left[\frac{2^{(1-\alpha)k}}{n}\right]$$

uniformly in x . Thus the integral (2) is

$$O\left[\frac{1}{n} \sum_{k=1}^m 2^{(1-\alpha)k}\right], \quad m = \left[\frac{\log n}{\log 2}\right],$$

that is to say

$$O\left[\frac{1}{n} 2^{(1-\alpha)m}\right] = O(n^{-\alpha})$$

which proves Theorem I. Clearly if we start from the assumption $f \in \text{lip } \alpha$ we prove in the same way $|s_n(x) - f(x)| = o(n^{-\alpha})$.

An easy consequence of Theorem I is that we have $|s_n(x) - f(x)| = O(n^{-\alpha})$ whenever $f(x)$ is continuous, of bounded variation, and its total variation $V(x)$ belongs to $\text{Lip } \alpha$ ($0 < \alpha < 1$).

(An elementary example of a continuous function of bounded variation whose total variation does not admit the same modulus of continuity as the function itself is provided by the function $x^\beta \sin 1/x$ ($1 < \beta < 2$) which belongs to $\text{Lip}(\beta/2)$ and whose total variation can not belong to $\text{Lip } \alpha$ for $\alpha > \beta - 1$. The proof is left to the reader.)

Proof of Theorem II. Let α be given ($0 < \alpha < 1$). For every positive integer n we consider the function continuous and of period 2π defined as follows

$$f_n(x) = \frac{1}{n^\alpha} \sin nx \quad \text{for} \quad 0 \leq x \leq \frac{[n^\alpha]\pi}{n},$$

$$f_n(x) = 0 \quad \text{for} \quad \frac{[n^\alpha]\pi}{n} \leq x \leq \pi,$$

$$f_n(-x) = f_n(x)$$

where $[n^\alpha]$ denotes as usual the integral part of n^α . We write $[n^\alpha]\pi/n = \lambda_n$.

The total variation of $f_n(x)$ in $(-\pi, +\pi)$ is

$$2 \int_0^{\lambda_n} n^{1-\alpha} |\cos nx| dx < 2n^{1-\alpha} \lambda_n < 2\pi.$$

Moreover, $f_n(x)$ belongs to $\text{Lip } \alpha$ uniformly in n . For, if x_1, x_2 belong to the interval $(0, \lambda_n)$, we have

$$f_n(x_2) - f_n(x_1) = \frac{1}{n^\alpha} 2 \sin n \frac{x_2 - x_1}{2} \cos n \frac{x_2 + x_1}{2};$$

thus, if $|x_2 - x_1| \geq 1/n$, we have

$$|f_n(x_2) - f_n(x_1)| \leq \frac{2}{n^\alpha} \leq 2 |x_2 - x_1|^\alpha.$$

On the other hand, if $|x_2 - x_1| < 1/n$, we have

$$|f_n(x_2) - f_n(x_1)| < \frac{2}{n^\alpha} \cdot \frac{n|x_2 - x_1|}{2} < \frac{|x_2 - x_1|}{|x_2 - x_1|^{1-\alpha}} = |x_2 - x_1|^\alpha.$$

Thus, taking into account that $f_n(x)$ is everywhere continuous, even, and vanishes in (λ_n, π) , we deduce that $|f_n(x_2) - f_n(x_1)| \leq 4|x_2 - x_1|^\alpha$, for any two points of the interval $(-\pi, \pi)$.

Finally, let us note that

$$\int_0^\pi f_n(t) \frac{\sin nt}{t} dt = \frac{1}{n^\alpha} \int_0^{\lambda_n} \frac{\sin^2 nt}{t} dt = \frac{1}{n^\alpha} \int_0^{[n^\alpha]\pi} \frac{\sin^2 v}{v} dv > A \frac{\log n}{n^\alpha},$$

A being a constant depending on α only, and that

$$\begin{aligned} \int_0^\pi f_n(t) \frac{\sin qt}{t} dt &= \frac{1}{n^\alpha} \int_0^{\lambda_n} \frac{\sin nt \sin qt}{t} dt \\ &= \frac{1}{n^\alpha} \int_0^{\lambda_n} \frac{\sin^2 (n+q)t/2}{t} dt - \frac{1}{n^\alpha} \int_0^{\lambda_n} \frac{\sin^2 (n-q)t/2}{t} dt \\ &= \frac{1}{n^\alpha} \int_{\lambda_n|n-q|/2}^{\lambda_n|n+q|/2} \frac{\sin^2 v}{v} dv > 0. \end{aligned}$$

Let now $\omega(n)$ be a function increasing infinitely with n as slowly as we please. Let $\{n_k\}$ be a sequence of integers such that

$$\sum \frac{1}{\omega(n_k)} < \infty.$$

Consider the function

$$\phi(x) = \sum_{k=1}^{\infty} \frac{f_{n_k}(x)}{\omega(n_k)}.$$

$\phi(x)$ is continuous, of period 2π , of bounded variation, and belongs to Lip α . If $s_n(x)$ denotes the n th partial sum of its Fourier series, we have

$$s_n(0) = \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin nt}{t} dt + O\left(\frac{1}{n}\right).$$

Hence, by the preceding remarks,

$$s_{n_k}(0) > \frac{A}{\pi} \frac{\log n_k}{\omega(n_k) \cdot n_k^\alpha} - O\left(\frac{1}{n_k}\right)$$

which is the same as

$$|s_{n_k}(0) - \phi(0)| > C \frac{\log n_k}{\omega(n_k) n_k^\alpha},$$

C being a positive constant, and thus proves our theorem.

2. In this section, which is not connected with the preceding one, we collect some other results concerning mainly the order of magnitude of $s_n - f$ and on the order of s_n , as compared with the corresponding expressions for the conjugate series.

THEOREM III. *Let $f(x)$ be a continuous function of period 2π , and $s_n(x)$ the partial sum of order n of its Fourier series. Let $\bar{f}(x)$ be the function conjugate to f , and $\bar{s}_n(x)$ its Fourier series partial sum of order n . If $f(x) - s_n(x) = O(n^{-\alpha})$ uniformly in x ($\alpha > 0$), then $\bar{f}(x) - \bar{s}_n(x) = O(n^{-\alpha})$ uniformly in x .*

Suppose first that $0 < \alpha < 1$. Then, by a well known theorem (see de la Vallée Poussin [3, p. 57]), $f(x)$ being approximable by trigonometric polynomials of order n with an error of order $n^{-\alpha}$, $f(x)$ belongs to Lip α . Hence, by Privaloff's theorem (see T.S. p. 156), $\bar{f}(x)$ belongs also to Lip α .

This implies, by S. Bernstein's theorem (see T.S. p. 62),

$$(1) \quad \sigma_n(x) - f(x) = O(n^{-\alpha}),$$

$$(2) \quad \bar{\sigma}_n(x) - \bar{f}(x) = O(n^{-\alpha})$$

uniformly in x , where σ_n and $\bar{\sigma}_n$ denote Fejér's sums of order n of $f(x)$ and $\bar{f}(x)$ respectively.

By (1) and by our hypothesis $s_n(x) - \sigma_n(x) = O(n^{-\alpha})$. Hence, by S. Bernstein's theorem on the maximum of the derivative of a trigonometric polynomial (see T.S. p. 155)

$$(3) \quad \frac{s'_n(x) - \sigma'_n(x)}{n+1} = O(n^{-\alpha}),$$

the primes denoting derivatives with respect to x .

Writing $s_n = A_0 + A_1 + \dots + A_n$, where $A_n = a_n \cos nx + b_n \sin nx$, and $\bar{s}_n = B_1 + B_2 + \dots + B_n$, where $B_n = a_n \sin nx - b_n \cos nx$, we have

$$s_n - \sigma_n = \frac{A_1 + 2A_2 + \dots + nA_n}{n+1}.$$

Hence (3) gives

$$(4) \quad \frac{B_1 + 2^2 B_2 + \dots + n^2 B_n}{(n+1)^2} = O(n^{-\alpha}).$$

Denoting now by $\bar{\sigma}_n^2$ the Cesàro mean of the second order of $\bar{f}$, we have

$$\bar{\sigma}_n = \frac{nB_1 + (n-1)B_2 + \dots + B_n}{n+1},$$

$$\bar{\sigma}_n^2 = \frac{n(n+1)B_1 + (n-1)nB_2 + \dots + 1 \cdot 2B_n}{(n+1)(n+2)},$$

so that

$$\bar{\sigma}_n - \bar{\sigma}_n^2 = \frac{nB_1 + 2(n-1)B_2 + \cdots + n \cdot 1 \cdot B_n}{(n+1)(n+2)}$$

and

$$\begin{aligned} \bar{\mathfrak{s}}_n - \bar{\sigma}_n - \frac{n+2}{n+1}(\bar{\sigma}_n - \bar{\sigma}_n^2) &= \frac{(n+1)B_1 + (n+1)2B_2 + \cdots + (n+1)nB_n}{(n+1)^2} \\ &\quad - \frac{nB_1 + (n-1)2B_2 + \cdots + n \cdot 1 \cdot B_n}{(n+1)^2} \\ &= \frac{B_1 + 2^2B_2 + \cdots + n^2B_n}{(n+1)^2} = O(n^{-\alpha}), \end{aligned}$$

by (4). But $\bar{\sigma}_n - \bar{f} = O(n^{-\alpha})$ obviously implies $\bar{\sigma}_n^2 - \bar{f} = O(n^{-\alpha})$, hence $\bar{\sigma}_n - \bar{\sigma}_n^2 = O(n^{-\alpha})$, and thus $\bar{\mathfrak{s}}_n - \bar{\sigma}_n = O(n^{-\alpha})$. Comparing this with (2) we see that

$$\bar{\mathfrak{s}}_n(x) - \bar{f}(x) = O(n^{-\alpha})$$

uniformly in x , which was to be proved.

Suppose now that $\alpha \geq 1$. We have, by hypothesis,

$$R_n = \sum_n A_p = O(n^{-\alpha}).$$

Let $\beta = \alpha - \epsilon$ where $0 < \epsilon < 1$. We have

$$\sum_n A_p p^\beta = \sum_n (R_p - R_{p+1}) p^\beta = R_n n^\beta + \sum_{n+1}^\infty R_p [p^\beta - (p-1)^\beta] = O(n^{-\epsilon}).$$

Hence, by the above proved result,

$$S_n = \sum_n B_p p^\beta = \sum_1^\infty B_p p^\beta - \sum_1^{n-1} B_p p^\beta = O(n^{-\epsilon}),$$

and this implies

$$\sum_n B_p = \sum_n \frac{S_p - S_{p+1}}{p^\beta} = \frac{S_n}{n^\beta} + \sum_{n+1}^\infty S_p \left[\frac{1}{p^\beta} - \frac{1}{(p-1)^\beta} \right] = O(n^{-\alpha}),$$

which completes the proof.

THEOREM IV. Let $f(x)$ and $s_n(x)$ have the same meaning as in Theorem III. If $f(x) - s_n(x) = O(n^{-\alpha})$, $0 < \alpha < 1$, uniformly in x , then the derivative $s'_n(x)$ is $O(n^{1-\alpha})$ uniformly in x .

We have, as in the proof of Theorem III, and using the same notations:

$$s_n(x) - \sigma_n(x) = O(n^{-\alpha})$$

which implies

$$s'_n(x) - \sigma'_n(x) = O(n^{1-\alpha}).$$

But

$$\begin{aligned}\sigma'_n(x) &= - \frac{nB_1 + 2(n-1)B_2 + \cdots + nB_n}{n+1} \\ &= - (n+2)[\bar{\sigma}_n - \bar{\sigma}_n^2] = O(n^{1-\alpha}),\end{aligned}$$

since we have seen that $\bar{\sigma}_n - \bar{f}$ and $\bar{\sigma}_n^2 - \bar{f}$ are $O(n^{-\alpha})$. Hence also

$$s'_n(x) = O(n^{1-\alpha}).$$

A proof analogous to that of Theorem III shows that, if $f \in L$, and if $\int_0^{2\pi} |f - s_n| dx = O(n^{-\alpha})$, then $\int_0^{2\pi} |\bar{f} - \bar{s}_n| dx = O(n^{-\alpha})$. Instead of Bernstein's theorem on the derivatives of trigonometric polynomials we apply the corresponding result for mean values (T.S. p. 155). It may be added that for $p > 1$ the inequality $\{\int_0^{2\pi} |f - s_n|^p dx\}^{1/p} = O(n^{-\alpha})$ implies $\{\int_0^{2\pi} |\bar{f} - \bar{s}_n|^p dx\}^{1/p} = O(n^{-\alpha})$ and that this may be obtained by a similar argument. The result, however, is of no particular interest, being a corollary of the well known inequality $\int_0^{2\pi} |\bar{g}|^p dx \leq M_p^2 \int_0^{2\pi} |g|^p dx$ of M. Riesz (T.S. p. 147).

A similar extension applies to Theorem IV.

THEOREM V. Let $\sum_0^\infty A_n$ be any trigonometric series ($A_n = a_n \cos nx + b_n \sin nx$), and let $\sum_1^\infty B_n$ be the conjugate series. Denote by s_n and $\bar{s}_n$ the partial sums $\sum_0^n A_k$ and $\sum_1^n B_k$ respectively. Let $\alpha > 0$. If $s_n(x) = O(n^\alpha)$ uniformly in x , then $\bar{s}_n(x) = O(n^\alpha)$ uniformly in x . If $\int_0^{2\pi} |s_n| dx = O(n^\alpha)$, then $\int_0^{2\pi} |\bar{s}_n| dx = O(n^\alpha)$.

This theorem is analogous to a well known result stating that, if $z = re^{i\theta}$, $\alpha > 0$, and if $f(z) = u(r, \theta) + iv(r, \theta)$ is regular for $|z| < 1$, then $u(r, \theta) = O(1-r)^{-\alpha}$ implies $v(r, \theta) = O(1-r)^{-\alpha}$ and $\int_0^{2\pi} |u(r, \theta)| d\theta = O(1-r)^{-\alpha}$ implies $\int_0^{2\pi} |v(r, \theta)| d\theta = O(1-r)^{-\alpha}$ (see Hardy and Littlewood [2], Cartwright [1], or Zygmund [4]).

Suppose that

$$|s_n(x)| \leq \phi(n)$$

for all x and n . Then, by S. Bernstein's theorem,

$$\left| \frac{s'_n(x)}{n} \right| \leq \phi(n).$$

Denoting by $\bar{\sigma}_n$ the Cesàro means of order 1 of $\sum_1^\infty B_n$, we have

$$(5) \quad |\bar{\sigma}_n - \bar{s}_n| = \left| \frac{s'_n(x)}{n+1} \right| \leq \phi(n).$$

On the other hand,

$$|\bar{\sigma}_n - \bar{\sigma}_{n-1}| = \left| \frac{s'_n(x)}{n(n+1)} \right| \leq \frac{\phi(n)}{n+1}.$$

Hence

$$|\bar{\sigma}_n| \leq |\bar{\sigma}_1| + |\bar{\sigma}_2 - \bar{\sigma}_1| + \cdots + |\bar{\sigma}_n - \bar{\sigma}_{n-1}| \leq \sum_1^n \frac{\phi(\nu)}{\nu+1} = O(n^\alpha),$$

if $\phi(n) = O(n^\alpha)$. And this, compared with (5), gives $\bar{s}_n(x) = O(n^\alpha)$, the required result.

The second part of Theorem V is proved similarly.

Theorem V is a special case of the following theorem.

THEOREM VI. Let $\alpha > 0$, $\beta > -1$. Let $\sigma_n^\beta(x)$ and $\bar{\sigma}_n^\beta(x)$ denote respectively the (C, β) means of the series $\sum_0^\infty A_k$ and $\sum_1^\infty B_k$. If $\sigma_n^\beta(x) = O(n^\alpha)$ uniformly in x , then $\bar{\sigma}_n^\beta(x) = O(n^\alpha)$ uniformly in x . If $\int_0^{2\pi} |\sigma_n^\beta(x)| dx = O(n^\alpha)$, then $\int_0^{2\pi} |\bar{\sigma}_n^\beta(x)| dx = O(n^\alpha)$.

Let γ_n^β denote the Cesàro numbers, $\gamma_n^\beta = (\beta+1)(\beta+2) \cdots (\beta+n)/n!$. The Cesàro sums σ_n^β for any numerical series $u_0 + u_1 + u_2 + \cdots$ are defined by the formula

$$\sigma_n^\beta = \frac{1}{\gamma_n^\beta} \sum_{r=0}^n \gamma_{n-r}^\beta u_r.$$

It is easy to verify that

$$\begin{aligned} \sigma_n^\beta - \sigma_n^{\beta+1} &= \frac{1}{n+\beta+1} \frac{1}{\gamma_n^\beta} \sum_{r=0}^n \gamma_{n-r}^\beta \nu u_r, \\ \sigma_n^\beta - \sigma_{n-1}^\beta &= \frac{\beta}{n(n+\beta)} \frac{1}{\gamma_{n-1}^{\beta-1}} \sum_{r=0}^n \gamma_{n-r}^{\beta-1} \nu u_r. \end{aligned}$$

In particular, the (C, β) means of the series $\sum_0^\infty A_k$ and $\sum_1^\infty B_k$ satisfy the equations

$$\begin{aligned} \bar{\sigma}_n^\beta(x) - \bar{\sigma}_n^{\beta+1}(x) &= -\frac{1}{n+\beta+1} \frac{d}{dx} \sigma_n^\beta(x), \\ \bar{\sigma}_n^\beta(x) - \bar{\sigma}_{n-1}^\beta(x) &= -\frac{\beta}{n(n+\beta)} \frac{d}{dx} \bar{\sigma}_n^{\beta-1}(x). \end{aligned}$$

If we use these two equations, the proof of Theorem VI becomes parallel to that of Theorem V.

It is clear that in all the results the " O " may be replaced by " o ."

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